Change of the coordinate system and the rotation about an arbitrary line in $\mathbb{R}^3$

Problem 1: We are working in the Cartesian coordinate system $\langle O, \mathbf{e}_0, \mathbf{e}_1 \rangle$, where a point X has coordinates

$$X = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Tell the coordinates of the point X in a coordinate system $\langle A, B - A, C - A \rangle$, where

$$A = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad C = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Solution: First start by recalling, that every coordinate system in plane is defined by its origin – a point – and the two vectors. This way we can record the old, standard Cartesian coordinate system as follows:

$$\langle O, \mathbf{e}_0, \mathbf{e}_1 \rangle = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle. \tag{1}$$

Note, that we are using augmented vectors for representing points and vectors in our computations. By simple computation and following the same fashion, we are allowed to write the other, new coordinate system

$$\langle A, B - A, C - A \rangle = \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle. \tag{2}$$

Both coordinate systems, together with the point X are depicted in the figure 1.

Now our problem can rephrased. Let denote by $X_s = (2, 1, 1)^\top$ the point X in the old coordinate system and by X_n the point X in the new coordinate system. Our task is now to find the coordinates of X_n in the new coordinate system. This can be done by using the simple fact from linear algebra, known as **change of basis**, which says following:

$$X_s = \mathbb{M}_n X_n, \tag{3}$$

where the matrix $\mathbb{M}_n$ is created by the column vectors of the new basis (beware of the order of the columns), i.e.

$$\mathbb{M}_n = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Equation (3) yields that the coordinates of X_n are computed as

$$X_n = \mathbb{M}_n^{-1} X_s. \tag{5}$$

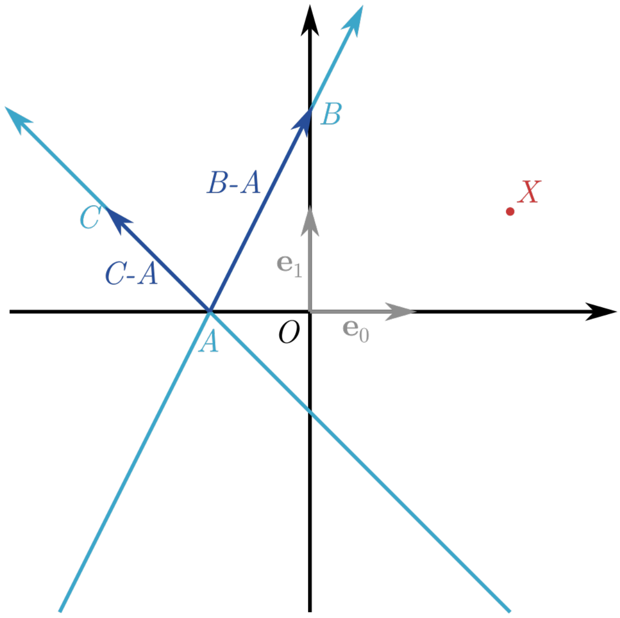

Figure 1: The old (Cartesian) coordinate system is black, the new coordinate system is depicted in blue and the point X in red.

At this points we see, that X_n can be computed only if $\mathbb{M}_n$ is invertible (otherwise the coordinate system would be degenerate). This is why it is useful to compute $\mathbb{M}_n^{-1}$ using the adjugate matrix method, because firstly we have to compute the $\det(\mathbb{M}_n)$, which tells us directly about the invertibility.

After the necessary operations we can conclude, that

$$X_n = \begin{pmatrix} {}^{1}/_{3} & {}^{1}/_{3} & {}^{1}/_{3} \\ {}^{-2}/_{3} & {}^{1}/_{3} & {}^{-2}/_{3} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{4}/_{3} \\ {}^{-5}/_{3} \\ 1 \end{pmatrix}. \tag{6}$$

The geometric meaning behind the coordinates of X_s and X_n can be found in the figure 2.

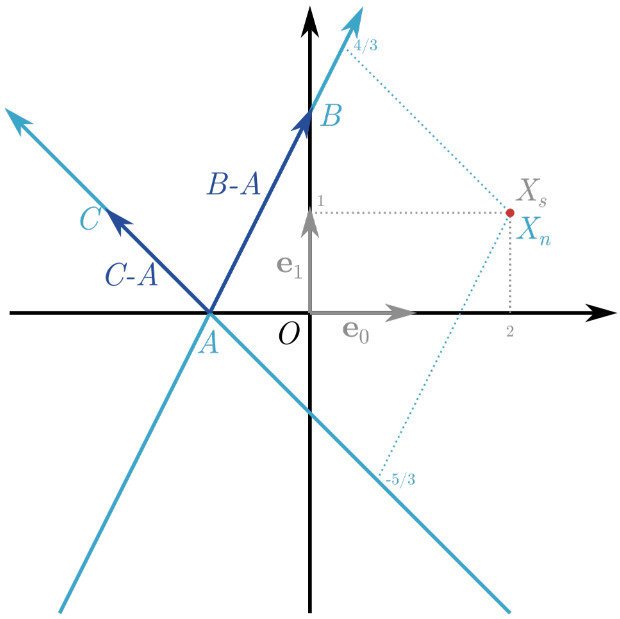

Figure 2: The coordinates of X_s in the old coordinate system are depicted in gray, the coordinates of X_n in the new coordinate system are depicted in blue.

Problem 2: Consider a point

$$X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and a line p defined by a point P and a vector $\mathbf{u}$ with coordinates

$$P = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

What are the coordinates of a point $\hat{X}$, which is the result of the rotation of the point X about the line p by the angle $\varphi = \frac{2}{3}\pi$? Perform the computation using

- quaternions
- affine transformations

Solution: So far we have not talked about the rotations about the arbitrary line in $\mathbb{R}^3$. This kind of rotation is also an affine transformation, because it can be composed by translations and rotations about the coordinate axes, which are affine transformations too (recall the fact, that the composition of affine transformations is again an affine transformation).

The process of the rotation $\mathbb{R}_{p,\varphi}$ of the point X about the line p by the angle φ can be described in the several steps:

1. Translate p, so it passes through the origin, i.e. perform the translation by the vector $O - P$ (so the point P is identified with the origin O, see fig. 3 (b)).
2. The line p now creates the angle ψ with the axis z, see fig. 3 (c).
3. By the orthogonal projection of the point $R = O + \mathbf{u}$ onto the plane xy we obtain the point R', so we can easily obtain the angle θ, as can be seen again in the fig. 3 (c).
4. Now we can rotate the line p by the angle $-\theta$ about the z-axis, so it lies in the plane xz, see fig. 3 (d). Note, that X does not have to lie in the plane xz necessarily.
5. Rotate the line p about the y-axis by the angle $-\psi$, so it is finally identified with the axis z, see fig. 3 (e).
6. At this moment we can perform the rotation about the z-axis by the given angle φ, as we can see in the fig. 3 (e).
7. However, the result is still not our desired point $\hat{X}$. **We need to perform the inverse transformations,** so the line p moves to its original position.

Then, the rotation $\mathbb{R}_{p,\varphi}$ can be written as follows:

$$\mathbb{R}_{p,\varphi} = \mathbb{T}_{P-O} \circ \mathbb{R}_{z,\theta} \circ \mathbb{R}_{y,\psi} \circ \mathbb{R}_{z,\varphi} \circ \mathbb{R}_{y,-\psi} \circ \mathbb{R}_{z,-\theta} \circ \mathbb{T}_{O-P}, \tag{7}$$

where $\mathbb{T}_*$ denotes a translation by the given vector, and $\mathbb{R}_{*,*}$ denotes the rotation about the given axis by the given angle.

Several notes about the equation (7):

- The inverse transformation to the translation by the vector $\mathbf{v}$ is the translation by the vector $-\mathbf{v}$, and the inverse rotation about the given axis by the angle α is the rotation about the same axis by the angle $-\alpha$.
- The inverse transformations need to be performed in the opposite order as the "forward" transformations, e.g. if we first perform the translation $\mathbb{T}_{O-P}$, the inverse translation $\mathbb{T}_{P-O}$ is performed as the last one.
- The first performed transformation is the translation $\mathbb{T}_{O-P}$, i.e. we write the transformations from right to left.

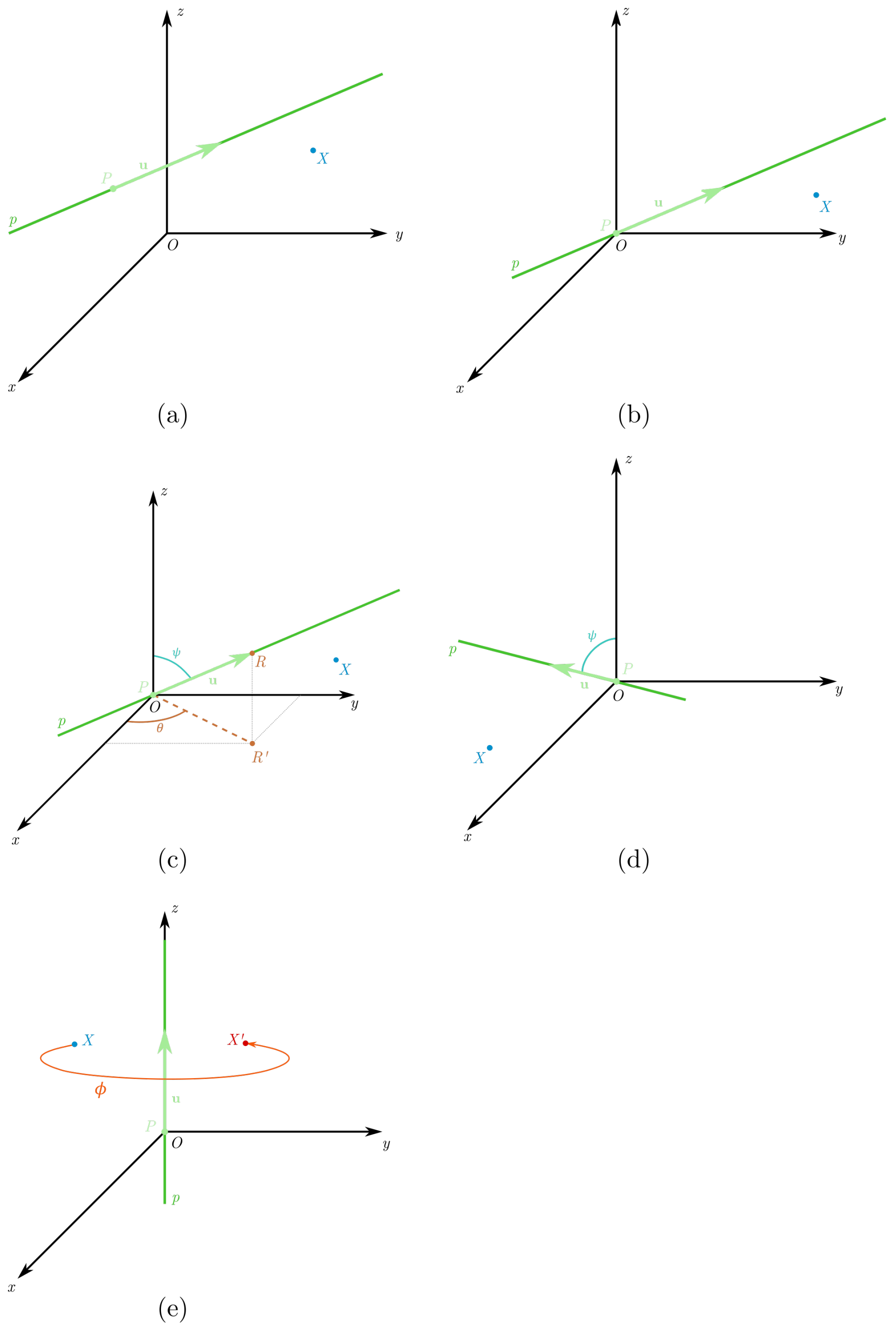

Figure 3: The process of rotation. Note, that in (c) we are actually finding the polar coordinates of the point R. The depicted situation does not reflect the problem which is solved in our case.

Finally, we are able to compute the desired point $\hat{X}$ as

$$\hat{X} = \mathbb{R}_{p,\varphi} X. \tag{8}$$

Now look, at our problem, where we will use augmented coordinates. As described in the process, firstly we need to compute the vector

$$O - P = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 0 \end{pmatrix}. \tag{9}$$

The we compute the coordinates of the point X', which is obtained by the translation of the point

X by the vector $O - P$ (X' is depicted as a point X in the fig. 3 (b)):

$$X' = \mathbb{T}_{O-P} X = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}. \tag{10}$$

The projection of the point $R = O + \mathbf{u} = (1, -1, 1, 1)^\top$ onto the plane xy is created simply by setting the z-coordinate to zero, thus the coordinates of the point R' are $(1, -1, 0, 1)^\top$. Now we need to determine the angles θ and ψ. The angle θ is created, by the vector $R' - O = (1, -1, 0, 0)^\top$ and the basis vector $\mathbf{e}_x = (1, 0, 0, 0)^\top$, which determines the x-axis. The computation of the angle between these two vectors is

$$\sphericalangle R' - O, \mathbf{e}_x = \arccos \frac{\langle R' - O, \mathbf{e}_x \rangle}{\|R' - O\| \|\mathbf{e}_x\|} = \arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}. \tag{11}$$

However, this angle is not the angle θ. The problem is, that the given formula always return a value in the interval $\langle 0, \pi \rangle$, i.e. only the angles which "lie" in the half-plane determined by the x-axis and the positive y-coordinates. But R' has a negative y-coordinate, i.e. it lies in the opposite half-plane. This is why we need to add π to the computed angle, i.e.

$$\theta = \frac{7\pi}{4}.$$

This is also in agreement with the fact, that if we represent the point R in spherical coordinates, the angle θ attains value in the range $\langle 0, 2\pi)$. Similarly, the angle $\psi \in \langle 0, \pi \rangle$ according to the spherical coordinates, so we can compute this angle as

$$\psi = \sphericalangle R - O, \mathbf{e}_z = \arccos \frac{\langle R - O, \mathbf{e}_z \rangle}{\|R - O\| \|\mathbf{e}_z\|} = \arccos \frac{\sqrt{3}}{3}. \tag{12}$$

Finally, we have all the information for computing the point $\hat{X}$:

$$\hat{X} = (\mathbb{T}_{P-O} \circ \mathbb{R}_{z,\theta} \circ \mathbb{R}_{y,\psi} \circ \mathbb{R}_{z,\varphi} \circ \mathbb{R}_{y,-\psi} \circ \mathbb{R}_{z,-\theta} \circ \mathbb{T}_{O-P}) X, \tag{13}$$

$$\begin{aligned} \hat{X} = & \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(^{7\pi}/_4) & -\sin(^{7\pi}/_4) & 0 & 0 \\ \sin(^{7\pi}/_4) & \cos(^{7\pi}/_4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & \begin{pmatrix} \cos\arccos(^{\sqrt{3}}/_3) & 0 & \sin\arccos(^{\sqrt{3}}/_3) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\arccos(^{\sqrt{3}}/_3) & 0 & \cos\arccos(^{\sqrt{3}}/_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(^{2\pi}/_3) & -\sin(^{2\pi}/_3) & 0 & 0 \\ \sin(^{2\pi}/_3) & \cos(^{2\pi}/_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & \begin{pmatrix} \cos\arccos(-^{\sqrt{3}}/_3) & 0 & \sin\arccos(-^{\sqrt{3}}/_3) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\arccos(-^{\sqrt{3}}/_3) & 0 & \cos\arccos(-^{\sqrt{3}}/_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-^{7\pi}/_4) & -\sin(-^{7\pi}/_4) & 0 & 0 \\ \sin(-^{7\pi}/_4) & \cos(-^{7\pi}/_4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}. \end{aligned} \tag{14}$$

If we are going to solve the same problem using quaternions, we are going to use the standard coordinates of points and vectors and keep in mind the rules listed in the lecture, especially:

- quaternion multiplication is **not** commutative,

- quaternion multiplication works in the same way as expansion of terms in $\mathbb{R}$,
- quaternion addition is performed by elements.

Quaternion spatial rotation works only in the case, when the line is passing through the origin, i.e. similarly as in the previous case, we need to translate the point X by the vector $O - P = (1, -2, 1)^\top$. Our result is the point $X' = (2, -2, 0)^\top$.

Now that the line is passing through the origin, we need to normalize the determining vector $\mathbf{u}$, i.e. to define the vector

$$\mathbf{u}_n = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \Big(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\Big)^\top. \tag{15}$$

A quaternion $\mathbf{q_u}$ can be assigned to the vector $\mathbf{u}_n$ as follows:

$$\mathbf{q_u} = \frac{1}{\sqrt{3}}\mathbf{i} + \Big(-\frac{1}{\sqrt{3}}\Big)\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}. \tag{16}$$

Similarly, this can be done for the point X', i.e. we can assign the following quaternion to it:

$$\mathbf{x} = 2\mathbf{i} - 2\mathbf{j} + 0\mathbf{k}. \tag{17}$$

Note, that when using quaternions, we are making no difference between points and vectors, compared to the augmented coordinates.

The rotation by the angle $\varphi = {}^{2\pi}/_3$ has the following expression in the world of quaternions:

$$\mathbf{q}_R = \cos\frac{\varphi}{2} + \mathbf{q_u}\sin\frac{\varphi}{2} = \frac{1}{2} + \Big(\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}\Big). \tag{18}$$

Now the result $\tilde{\mathbf{x}}$ of the rotation $\mathbf{q}_R$ of the quaternion $\mathbf{x}$ is given by

$$\tilde{\mathbf{x}} = \mathbf{q}_R\,\mathbf{x}\,\mathbf{q}_R^{-1}. \tag{19}$$

As we can see, we need to use the inverse of $\mathbf{q}_R$, but this is nothing else, than just performing the rotation in the opposite direction, what is easily expressed as

$$\mathbf{q}_R^{-1} = \cos\frac{\varphi}{2} - \mathbf{q_u}\sin\frac{\varphi}{2} = \frac{1}{2} - \Big(\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}\Big). \tag{20}$$

Now, by the direct computation with respect to the quaternion rules we compute the quaternion

$$\tilde{x} = \mathbf{q}_R\,\mathbf{x}\,\mathbf{q}_R^{-1} = \Big(\frac{1}{2} + \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}\Big)\Big(2\mathbf{i} - 2\mathbf{j}\Big)\Big(\frac{1}{2} - \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}\Big) = 2\mathbf{i} + 2\mathbf{k}, \tag{21}$$

which represents the point $\tilde{X} = (2, 0, 2)^\top$.

However, the point $\tilde{X}$ is still not our desired result, because at the beginning we translated the line to the origin. This means, we need to perform the backward translation, i.e. by the vector $P - O = (-1, 2, 1)^\top$ and this operation finally gives us the final result

$$\hat{X} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{22}$$