Problems to solve during the (out)break

PART 1 – STEP-BY-STEP SOLUTIONS

Problem 1: *Lambertian reflectance* says, that the light intensity on an ideally diffuse surface is directly proportional to cosine of the angle β, which is created by the normal vector of the surface $\mathbf{n}$ and the incoming light ray $\mathbf{s}$.
Calculate the light intensity in the point $P = (0, 10, 0)^\top$, which lies on a surface with the normal vector $\mathbf{n} = (0, 1, 0)^\top$ and the coordinates of the light source are $S = (20, 20, 40)^\top$.

The Lambertian reflectance assumes, that the object is **ideally diffuse**, i. e. for all its points, the reflectance is ideally diffuse. In practice, this means, that the light is reflected in all directions with same probability.

In real world, ideally diffuse materials do not exist. However, as we will see later, the diffuse reflection is a part of the more advanced shading model.

Using this type of reflectance, the light intensity is computed as

$$\cos\beta = \mathbf{n} \cdot \mathbf{s}, \tag{1}$$

where $\mathbf{n}$ is the normal vector at the point, where the light ray hits the surface, and $\mathbf{s}$ is the direction vector of the ray. Note, that the vector starts on the point of the surface, not the point defining the light source. Also, the equation (1) assumes the vectors to be unit.

Using the properties of the cosine function we may formulate the following properties:

- The light intensity is **maximal**, i. e. $\mathbf{n} \cdot \mathbf{s} = 1$, if the respective vectors are linearly dependent (the light source is placed directly above the given point).
- The light intensity is **minimal**, i. e. $\mathbf{n} \cdot \mathbf{s} = 0$, if the respective vectors are perpendicular (the light source shines along the surface in the given point).

However, the cosine function may attain negative values, which means, the light source is placed below the surface, i. e. it does not shine on the visible side of the object. In this case, the light intensity is usually set to 0.

In the practical example we need to construct the light-direction vector $\mathbf{s}$, s. t. $\|\mathbf{s}\| = 1$:

$$\mathbf{s} = \frac{S - P}{\|S - P\|} = \frac{(20, 10, 40)^\top}{\sqrt{20^2 + 10^2 + 40^2}} = \frac{1}{\sqrt{21}}(2, 1, 4)^\top. \tag{2}$$

It is obvious that $\|\mathbf{n}\| = 1$ so we may proceed to the computation of the light intensity in the point P:

$$\cos\beta = \mathbf{n} \cdot \mathbf{s} = \frac{1}{\sqrt{21}}(2 \cdot 0 + 1 \cdot 1 + 4 \cdot 0) = \frac{1}{\sqrt{21}} \approx 0.218. \tag{3}$$

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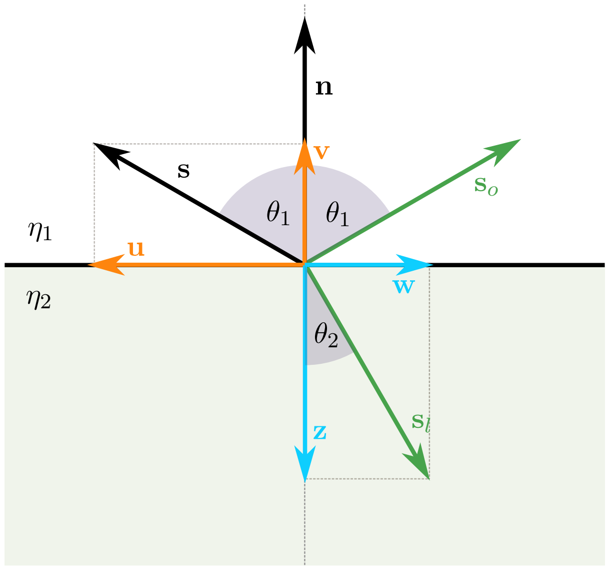
In practice, this information tells us that the color intensity in the point P (in RGB model the intensity of each color channel) is multiplied by the value 0.218 if the surface is shaded using the Lambertian reflectance.

Problem 2: Using the *Snell's law* $\eta_1 \sin\theta_1 = \eta_2 \sin\theta_2$, and the fact $\|\mathbf{s}\| = \|\mathbf{n}\| = 1$, prove that following equations for an ideally reflected and refracted ray hold:

$$\mathbf{s}_o = 2(\mathbf{s}\cdot\mathbf{n})\mathbf{n} - \mathbf{s}$$

$$\mathbf{s}_l = -\frac{\eta_1}{\eta_2}(\mathbf{s} - (\mathbf{s}\cdot\mathbf{n})\mathbf{n}) - \sqrt{1 - \left(\frac{\eta_1}{\eta_2}\right)^2 (1 - (\mathbf{s}\cdot\mathbf{n})^2)}\,\mathbf{n}$$



Firstly, let us introduce the notation:
$\mathbf{s}$ – incoming ray (note the direction),
$\mathbf{n}$ – normal vector,
$\mathbf{s}_o$ – reflected ray,
$\mathbf{s}_l$ – refracted ray,
θ_1 – angle of reflection,
θ_2 – angle of refraction,
η_1, η_2 – indices of refraction (IOR) for different media (each material has unique IOR).

Snell's law may be applied only to isotropic materials. The *isotropic materials* are those whose angle of reflection (the visual appearance) does not change under the rotation around the normal. The most of the common materials are isotropic, e. g. *glass, air*. On the contrary *anisotropic materials* are the ones whose angle of reflection changes under the rotation around the normal. The examples of anisotropic materials are *silk* or *wood*.

Since we need to find out the direction of the reflected and refracted ray, we construct them assuming $\|\mathbf{s}_o\| = \|\mathbf{s}_l\| = 1$.

To prove the first part we use the fact, that the vector $\mathbf{s}$ may be rewritten as $\mathbf{s} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u}, \mathbf{v}$ are such that

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 0, \\ \frac{\mathbf{n} \cdot \mathbf{v}}{\|\mathbf{n}\| \|\mathbf{v}\|} &= 1. \end{aligned} \tag{4}$$

The equation (4) says, that the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular and moreover, the vector $\mathbf{v}$ is the non-zero multiple of the vector $\mathbf{n}$. Note, that the vectors $\mathbf{v}$ and $\mathbf{n}$ are not identical, since $\|\mathbf{v}\| \neq 1$.

Since the angle of incidence and the angle of reflection are same, we can express the vector $\mathbf{s}_o$ as

$$\mathbf{s}_o = \mathbf{v} - \mathbf{u} = \mathbf{v} - (\mathbf{s} - \mathbf{v}) = 2\mathbf{v} - \mathbf{s}. \tag{5}$$

Using the fact, that the $\mathbf{n}$ is a unit vector, it is obvious that $\mathbf{v} = \|\mathbf{v}\|\mathbf{n}$. To get the value $\|\mathbf{v}\|$ we need to realize what the assumptions $\mathbf{s} = \mathbf{u} + \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{v} = 0$ mean geometrically. It means nothing else than the tips of the vectors $\mathbf{s}$ and $\mathbf{v}$ and the starting point create the right triangle.

This observation allows to express the cosine of θ_1 in two ways:

$$\begin{aligned} \cos\theta_1 &= \mathbf{s} \cdot \mathbf{n}, \\ \cos\theta_1 &= \frac{\|\mathbf{v}\|}{\|\mathbf{s}\|} = \|\mathbf{v}\|. \end{aligned} \tag{6}$$

Now it is obvious, that $\|\mathbf{v}\| = \mathbf{s} \cdot \mathbf{n}$. Having this value, the last step of the proof of the first part is to plug the expression of $\mathbf{v}$ into the equation (5).

To prove the second part we will use the similar approach as before. We rewrite the vector $\mathbf{s}_l$ as $\mathbf{s}_l = \mathbf{w} + \mathbf{z}$, where $\mathbf{w}, \mathbf{z}$ are such that

$$\begin{aligned} \mathbf{w} \cdot \mathbf{z} &= 0, \\ \frac{\mathbf{n} \cdot \mathbf{z}}{\|\mathbf{n}\| \|\mathbf{z}\|} &= -1. \end{aligned} \tag{7}$$

However, in this case we need to use the Snell's law, because the refracted ray passes through the different medium than the incoming ray. It is also useful to note, that the reflected ray passes through the same medium as the incoming ray. Thus it is independent of the material of the medium and indices of refraction are absent in the computation of its direction.

To compute the direction of the refracted ray we express the vectors $\mathbf{w}$ and $\mathbf{z}$ separately.

Using the fact, that the vectors $\mathbf{z}$ and $\mathbf{s}_l$ create the right triangle in the same sense as in the previous part, we may say the following:

$$\begin{aligned} \sin\theta_1 &= \frac{\|\mathbf{u}\|}{\|\mathbf{s}\|} = \|\mathbf{u}\|, \\ \sin\theta_2 &= \frac{\|\mathbf{w}\|}{\|\mathbf{s}_l\|} = \|\mathbf{w}\|. \end{aligned} \tag{8}$$

By plugging these expressions into the Snell's law we get that

$$\|\mathbf{w}\| = \frac{\eta_1}{\eta_2} \|\mathbf{u}\|. \tag{9}$$

The equation (9) actually describes the ratio of the lengths of the vectors $\mathbf{w}$ and $\mathbf{u}$. Since it is obvious, that $\mathbf{w}$ and $\mathbf{u}$ are linearly dependent and moreover, they have opposite directions, we may conclude that

$$\mathbf{w} = -\frac{\eta_1}{\eta_2} \mathbf{u}. \tag{10}$$

Recalling the previous part of the proof, that $\mathbf{u} = \mathbf{s} - \mathbf{v}$, we get

$$\mathbf{w} = -\frac{\eta_1}{\eta_2}(\mathbf{s} - (\mathbf{s} \cdot \mathbf{n})\mathbf{n}). \tag{11}$$

To compute the vector $\mathbf{z}$, we start with usage of the Pythagorean theorem, which yields

$$\|\mathbf{s}_l\|^2 = \|\mathbf{w}\|^2 + \|\mathbf{z}\|^2, \tag{12}$$

and allows us to express the length

$$\|\mathbf{z}\| = \sqrt{1 - \|\mathbf{w}\|^2}. \tag{13}$$

Using the similar consideration as in the equations (9) and (10), i. e. using the fact the that the vectors $\mathbf{z}$ and $\mathbf{n}$ are linearly dependent and have opposite directions, we come to conclusion that

$$\mathbf{z} = -\sqrt{1 - \|\mathbf{w}\|^2}\,\mathbf{n}. \tag{14}$$

Using the second equation of (8) and the Snell's law we get

$$\|\mathbf{w}\|^2 = \sin^2\theta_2 = \left(\frac{\eta_1}{\eta_2}\right)^2 \sin^2\theta_1 = \left(\frac{\eta_1}{\eta_2}\right)^2 (1 - \cos^2\theta_1) = \left(\frac{\eta_1}{\eta_2}\right)^2 (1 - (\mathbf{s} \cdot \mathbf{n})^2). \tag{15}$$

Now by inserting these expressions into the initial expression of $\mathbf{s}_l$ we may consider the proof of the second part to be completed.

Problem 3: Air has the refraction index equal to 1.00026 and for water it is 1.33. Consider the ray passing from the water into the air. Compute the critical angle for the water, i.e. when the total internal reflections occurs.

The total internal reflection (TIR) (broadly said, there is no refraction) occurs only if the light travels from the medium with higher IOR to the medium with lower IOR.

The critical angle is dependent on the feasible values of the sine of the respective angle.

Let us denote by $\eta_1 = 1.33$ the IOR of water and by $\eta_2 = 1.00026$ the IOR of air. Since the ray is travelling from water, i. e. from the medium with higher IOR to air (the medium with lower IOR), the TIR may occur.

Finding the critical angle means to find such angle θ_1, so the angle of refraction $\theta_2 = \pi/2$. In other words, the "refracted" ray does not proceed to air, but glides along the boundary which separates the two media, that is, it is perpendicular to the normal vector in the given point.

By modification of the Snell's law we get the angle θ_1:

$$\theta_1 = \arcsin\left(\frac{\eta_2}{\eta_1}\sin\theta_2\right) = \arcsin\left(\frac{1.00026}{1.33}\sin\frac{\pi}{2}\right) \approx \arcsin 0.7521 \approx 48.77°. \tag{16}$$

The interpretation of this value says, that if the incidence angle $\theta_1 \geq 48.77°$, then the TIR occurs (i. e. no refraction). To illustrate this, let us consider the value $\theta_1 = \pi/3$ and let us compute the angle of refraction:

$$\theta_2 = \arcsin\left(\frac{\eta_1}{\eta_2}\sin\theta_1\right) = \arcsin\left(\frac{1.33}{1.00026}\sin\frac{\pi}{3}\right) \approx \arcsin 1.152 \rightsquigarrow \text{undefined}. \tag{17}$$

As we see in the equation (17), the angle θ_2 can not be computed, since the arcsine function is undefined for the given value, that is for the angle of incidence $\theta_1 = \pi/3$ the incoming ray is only reflected, never refracted.

Problem 4: Consider a light ray which strikes the glass plate (with constant non-zero thickness) placed in the vacuum. We know, that the velocity of light distribution in the glass is 54% of the one for vacuum.
What is the angle of refraction when the ray strikes the glass plate at an angle 60°? What is the angle of refraction, when the ray leaves the glass plate?

The Snell's law may be reformulated in terms of velocity of light distribution in certain medium. Since a refractive index describes how fast the light travels through the medium, it may be expressed using the velocity as

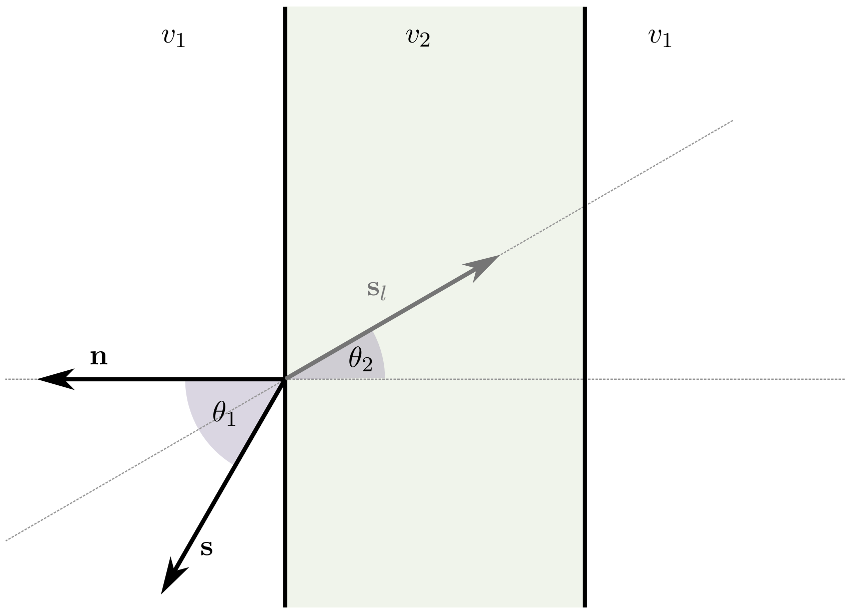
$$\eta = \frac{c}{v}, \tag{18}$$

where η denotes the IOR and v denotes the phase velocity of light distribution in the medium. The constant c is the speed of light in vacuum.

By inserting the equation (18) into the Snell's law we get

$$\begin{aligned} \eta_1 \sin\theta_1 &= \eta_2 \sin\theta_2 \\ \frac{c}{v_1}\sin\theta_1 &= \frac{c}{v_2}\sin\theta_2 \\ v_2 \sin\theta_1 &= v_1 \sin\theta_2. \end{aligned} \tag{19}$$

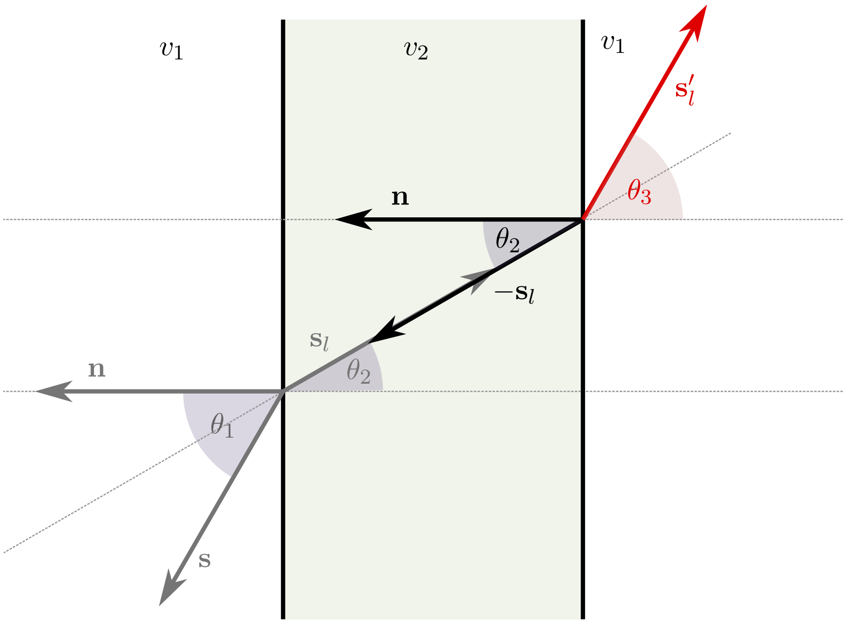
Now let us denote the velocity in the vacuum by v_1 and in the glass by v_2. As noted in the instructions, we know that $v_2 = 0.54 v_1$ and $\theta_1 = 60°$.



To get the first angle of refraction, denoted by θ_2 we just need to plug the known values into the modified Snell's law in (19). More precisely

$$\theta_2 = \arcsin\left(\frac{v_2}{v_1}\sin\theta_1\right) = \arcsin\left(\frac{0.54 v_1}{v_1}\sin 60°\right) \approx \arcsin 0.4677 \approx 27,882°. \tag{20}$$

Now, to compute the second angle of refraction θ_3 we use the fact, that the vector of incident ray is $-\mathbf{s}_l$ and the normal $\mathbf{n}$ at the point of intersection with surface is same as in the previous case, since we are talking about the glass plate of constant thickness.



This yields, that the angle of incidence is the angle θ_2 and now again by using the same approach we obtain that the angle θ_3 is given by

$$\theta_3 = \arcsin\left(\frac{v_1}{v_2}\sin\theta_2\right) = \arcsin\left(\frac{v_1}{0.54v_1}\sin 27,882^\circ\right) \approx \arcsin 0.866 \approx 60^\circ. \qquad (21)$$

We see, that $\theta_3 = \theta_1$ and moreover $\mathbf{s}'_l = -\mathbf{s}$. These two relations are generally true only if the normal vector $\mathbf{n}$ does not change, when passing from one media into another.